

Differential Equations Driven by Gaussian Signals II

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Abstract

Large classes of multi-dimensional Gaussian processes can be enhanced with stochastic Lévy area(s). In a previous paper, we gave sufficient and essentially necessary conditions, only involving variational properties of the covariance. Following T. Lyons, the resulting lift to a "Gaussian rough path" gives a robust theory of (stochastic) differential equations driven by Gaussian signals with sample path regularity worse than Brownian motion.

The purpose of this sequel paper is to establish convergence of Karhunen-Loeve approximations in rough path metrics. Particular care is necessary since martingale arguments are not enough to deal with third iterated integrals. An abstract support criterion for approximately continuous Wiener functionals then gives a description of the support of Gaussian rough paths as the closure of the (canonically lifted) Cameron-Martin space.

1 Introduction

Let X be a real-valued centered Gaussian process on $[0, 1]$ with continuous sample paths and (continuous) covariance $R = R(s, t) = \mathbb{E}(X_s X_t)$. We say that R has finite ρ -variation of R , in symbols $R \in C^{\rho-var}([0, 1]^2, \mathbb{R})$ if

$$\sup_D \sum_{i,j} |\mathbb{E}[(X_{t_{i+1}} - X_{t_i})(X_{t_{j+1}} - X_{t_j})]|^\rho < \infty.$$

An first consequence of this [10] is that (2D) ρ -variation of the covariance implies (the usual 1D) ρ -variation regularity of elements in the associated *Cameron-Martin space* \mathcal{H} , viewed as subspace of $C([0, 1], \mathbb{R})$. Good examples to have in mind are standard Brownian motion with $\rho = 1$ and fractional Brownian

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motion with Hurst parameter $H \in (0, 1/2]$ for which $\rho = 1/(2H)$. We then consider a d -dimensional, continuous, centered Gaussian process with independent components,

$$X = (X^1, \dots, X^d),$$

with respective covariances $R_1, \dots, R_d \in C^{\rho-var}$. Assuming momentarily smooth sample paths we can define $\mathbf{X} \equiv S_3(X)$ by its coordinates in the three "tensor-levels", $\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d$ and $\mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d$, obtained by iterated integration

$$\mathbf{X}^i = \int_0^\cdot dX_r^i, \quad \mathbf{X}^{i,j} = \int_0^\cdot \int_0^s dX_r^i dX_s^j, \quad \mathbf{X}^{i,j,k} = \int_0^\cdot \int_0^t \int_0^s dX_r^i dX_s^j dX_t^k,$$

with indices $i, j, k \in \{1, \dots, d\}$. It turns out that for $\rho \in [1, 2)$ all these integrals make sense as L^2 limits (similar to Itô's theory) and we cite the following

Theorem 1 ([10]) *Let $X = (X^1, \dots, X^d), Y = (Y^1, \dots, Y^d)$ be two continuous, centered jointly Gaussian processes defined on $[0, 1]$ such that (X^i, Y^i) is independent of (X^j, Y^j) when $i \neq j$. Let $\rho \in [1, 2)$ and assume the covariance of (X, Y) is of finite ρ -variation,*

$$|R_{(X,Y)}|_{\rho-var;[0,1]^2} \leq K < \infty.$$

Let $p > 2\rho$ and \mathbf{X}, \mathbf{Y} denote the natural lift¹ of X, Y respectively. Then there exist positive constants $\theta = \theta(p, \rho)$ and $C = C(p, \rho, K)$ such that

$$|d_{p-var}(\mathbf{X}, \mathbf{Y})|_{L^2} \leq C |R_{X-Y}|_{\infty;[0,1]^2}^\theta.$$

Let us consider a continuous, centered d -dimensional process W with independent components and finite $\rho \in [1, 2)$ -covariance and its piecewise linear approximations Z^n such that $Z_t \equiv Z_t^n$ for $t \in \{l/n : l = 0, \dots, n\}$. The above theorem, applied to $\mathbf{X} = S_3(Z^n), \mathbf{Y} = S_3(Z^m)$, shows that $S_3(Z^n)$ is Cauchy and this can be taken as definition of \mathbf{W} , the natural lift of W .

Of course, there should be nothing special about piecewise linear approximations² and we are indeed able to show that Karhunen-Loeve type approximations of form $S_3(\mathbb{E}(W|\mathcal{F}_n))$ converge to the same natural lift \mathbf{W} . Existence of Lévy's area for a multi-dimensional Gaussian process and regularity of Cameron-Martin paths are in fact closely related. Given a basis of \mathcal{H}_i , the Cameron-Martin space for $X^i, i \in \{1, \dots, d\}$, we have the L^2 - or Karhunen-Loeve expansion

$$X^i(\omega) = \sum_{k \in \mathbb{N}} Z_k^i(\omega) h^{i,k}.$$

If one assumes that $X = (X^1, \dots, X^d)$ lifts to \mathbf{X}

¹The present result in combination with a Cauchy argument effectively defines what is meant by natural lift.

²Observe how the assumptions of theorem 1 rule out McShane's famous example [12, Section on Approximations of the Wiener process].

- (i) having basic symmetry and integrability properties;
- (ii) via a Borel measurable map which coincides with the classical notion of iterated integrals on smooth paths;
- (iii) such that the lift of $X + h = (X^1 + h^1, \dots, X^d + h^d)$ coincides with the (rough path) translation of \mathbf{X} by h ;

then a martingale argument [4] shows that Lévy's area is given by (the anti-symmetric part of)

$$\mathbf{X}_t^{i,j} = \sum_{k,l} N_k^i N_l^j \int_0^t h_u^{i,k} dh_u^{j,l} \quad (1)$$

and a notion of iterated integral of elements in $\mathcal{H}_i, \mathcal{H}_j$ respectively is needed. With a view towards Young integrals, this underlines the importance of embedding the Cameron-Martin space into a ρ -variation path space. We will prove that formula (1) extends to the level of third iterated integrals and insist that martingale arguments alone are not enough for this and subtle correction terms need to be taken care of.

As was pointed out in [6] in the context of fractional Brownian motion, Karhunen-Loeve convergence + (iii), i.e. Cameron-Martin perturbations of the lift are given by rough path translation, imply a support description. In the generality of our discussion (iii) may fail. Careful revision of the arguments led us to an abstract support theorem for approximately continuous Wiener functionals; somewhat similar in spirit to Aida-Kusuoka-Stroock [1, Cor 1.13], cf. remark 12.

With a support description of Gaussian rough paths, one has typical rough path corollaries such as a Stroock-Varadhan type support theorem for solutions to rough differential equations

$$dY = V(Y) d\mathbf{X}(\omega), \quad V = (V_1, \dots, V_d)$$

and support description for $S_N(\mathbf{X})$, the canonically defined lift of \mathbf{X} to a process with values in the step- N free nilpotent group with d generators, for any $N \geq 3$. In contrast to the original motivation of Stroock-Varadhan, there is no Markovian structure here and hence no applications to maximum principles etc. Nonetheless, the support description of Gaussian rough path (and resultingly: of iterated integrals up to any given order) has been a key ingredient in establishing non-degeneracy of the Malliavin covariance matrix under Hörmander's condition [2].

The construction of a "natural" lift of a class of Gaussian processes containing fractional Brownian Motion with $H > 1/4$ is due to Coutin-Qian, [3] and the condition for H is optimal, [18, 4]. Support statements for lifted fractional Brownian Motion for $H > \frac{1}{3}$ are proved in [8], [6]; a Karhunen-Loeve type approximations for fractional Brownian Motion is studied in [19]. The present paper unifies and generalizes all these results.

1.1 Notations

Let (E, d) be a metric space and $x \in C([0, 1], E)$. It then makes sense to speak of α -Hölder- and p -variation "norms" defined as

$$\|x\|_{\alpha\text{-Hölder}} = \sup_{0 \leq s < t \leq 1} \frac{d(x_s, x_t)}{|t - s|^\alpha}, \quad \|x\|_{p\text{-var}} = \sup_{D=(t_i)} \left(\sum_i d(x_{t_i}, x_{t_{i+1}})^p \right)^{1/p}.$$

It also makes sense to speak of a d_∞ -distance of two such paths,

$$d_\infty(x, y) = \sup_{0 \leq t \leq 1} d(x_t, y_t).$$

Given a positive integer N the truncated tensor algebra of degree N is given by the direct sum

$$T^N(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus \dots \oplus (\mathbb{R}^d)^{\otimes N}.$$

With tensor product \otimes , vector addition and usual scalar multiplication, $T^N(\mathbb{R}^d) = (T^N(\mathbb{R}^d), \otimes, +, \cdot)$ is an algebra. Functions such as $\exp, \ln : T^N(\mathbb{R}^d) \rightarrow T^N(\mathbb{R}^d)$ are defined immediately by their power-series. Let π_i denote the canonical projection from $T^N(\mathbb{R}^d)$ onto $(\mathbb{R}^d)^{\otimes i}$. Let $p \in [1, 2)$ and $x \in C^{p\text{-var}}([0, 1], \mathbb{R}^d)$, the space of continuous \mathbb{R}^d -valued paths of bounded p -variation. We define $\mathbf{x} \equiv S_N(x) : [0, 1] \rightarrow T^N(\mathbb{R}^d)$ via iterated (Young) integration,

$$\mathbf{x}_t \equiv S_N(x)_t = 1 + \sum_{i=1}^N \int_{0 < s_1 < \dots < s_i < t} dx_{s_1} \otimes \dots \otimes dx_{s_i}$$

noting that $\mathbf{x}_0 = 1 + 0 + \dots + 0 = \exp(0) \equiv e$, the neutral element for \otimes , and that \mathbf{x}_t really takes values in

$$G^N(\mathbb{R}^d) = \{g \in T^N(\mathbb{R}^d) : \exists x \in C^{1\text{-var}}([0, 1], \mathbb{R}^d) : g = S_N(x)_1\},$$

a submanifold of $T^N(\mathbb{R}^d)$ and, in fact, a Lie group with product \otimes , called the free step- N nilpotent group with d generators. Because $\pi_1[\mathbf{x}_t] = x_t - x_0$ we say that $\mathbf{x} = S_N(x)$ is the canonical lift of x . There is a canonical notion of increments, $\mathbf{x}_{s,t} := \mathbf{x}_s^{-1} \otimes \mathbf{x}_t$. The dilation operator $\delta : \mathbb{R} \times G^N(\mathbb{R}^d) \rightarrow G^N(\mathbb{R}^d)$ is defined by

$$\pi_i(\delta_\lambda(g)) = \lambda^i \pi_i(g), \quad i = 0, \dots, N$$

and a continuous norm on $G^N(\mathbb{R}^d)$, homogenous with respect to δ , the *Carnot-Caratheodory norm*, is given

$$\|g\| = \inf \{\text{length}(x) : x \in C^{1\text{-var}}([0, 1], \mathbb{R}^d), S_N(x)_1 = g\}.$$

By equivalence of continuous, homogenous norms there exists a constant K_N such that

$$\frac{1}{K_N} \max_{i=1, \dots, N} |\pi_i(g)|^{1/i} \leq \|g\| \leq K_N \max_{i=1, \dots, N} |\pi_i(g)|^{1/i}.$$

The norm $\|\cdot\|$ induces a (left-invariant) metric on $G^N(\mathbb{R}^d)$ known as *Carnot-Caratheodory metric*, $d(g, h) := \|g^{-1} \otimes h\|$. Now let $x, y \in C_0([0, 1], G^N(\mathbb{R}^d))$, the space of continuous $G^N(\mathbb{R}^d)$ -valued paths started at the neutral element $\exp(0) = e$. We define α -Hölder- and p -variation distances

$$d_{\alpha-H\ddot{o}l}(\mathbf{x}, \mathbf{y}) = \sup_{0 \leq s < t \leq 1} \frac{d(\mathbf{x}_{s,t}, \mathbf{y}_{s,t})}{|t - s|^\alpha},$$

$$d_{p-var}(\mathbf{x}, \mathbf{y}) = \sup_{D=(t_i)} \left(\sum_i d(\mathbf{x}_{t_i, t_{i+1}}, \mathbf{y}_{t_i, t_{i+1}})^p \right)^{1/p}$$

and also the "0-Hölder" distance, locally $1/N$ -Hölder equivalent to $d_\infty(\mathbf{x}, \mathbf{y})$,

$$d_0(\mathbf{x}, \mathbf{y}) = d_{0-H\ddot{o}l}(\mathbf{x}, \mathbf{y}) = \sup_{0 \leq s < t \leq 1} d(\mathbf{x}_{s,t}, \mathbf{y}_{s,t}).$$

Note that $d_{\alpha-H\ddot{o}l}(\mathbf{x}, 0) = \|\mathbf{x}\|_{\alpha-H\ddot{o}l}$, $d_{p-var}(\mathbf{x}, 0) = \|\mathbf{x}\|_{p-var}$ where 0 denotes the constant path $\exp(0)$, or in fact, any constant path. The following path spaces will be useful to us

- (i) $C_0^{p-var}([0, 1], G^N(\mathbb{R}^d))$: the set of continuous functions \mathbf{x} from $[0, 1]$ into $G^N(\mathbb{R}^d)$ such that $\|\mathbf{x}\|_{p-var} < \infty$ and $\mathbf{x}_0 = \exp(0)$.
- (ii) $C_0^{0,p-var}([0, 1], G^N(\mathbb{R}^d))$: the d_{p-var} -closure of

$$\{S_N(x), x : [0, 1] \rightarrow \mathbb{R}^d \text{ smooth}\}.$$

- (iii) $C_0^{1/p-H\ddot{o}l}([0, 1], G^N(\mathbb{R}^d))$: the set of continuous functions \mathbf{x} from $[0, 1]$ into $G^N(\mathbb{R}^d)$ such that $d_{1/p-H\ddot{o}l}(0, \mathbf{x}) < \infty$ and $\mathbf{x}_0 = \exp(0)$.

- (iv) $C_0^{0,1/p-H\ddot{o}l}([0, 1], G^N(\mathbb{R}^d))$: the $d_{1/p-H\ddot{o}l}$ -closure of

$$\{S_n(x), x : [0, 1] \rightarrow \mathbb{R}^d \text{ smooth}\}.$$

Recall that a geometric p -rough path is an element of $C_0^{0,p-var}([0, 1], G^{[p]}(\mathbb{R}^d))$, and a weak geometric rough path is an element of $C_0^{p-var}([0, 1], G^{[p]}(\mathbb{R}^d))$. For a detail study of these spaces and their properties the reader is referred to [8].

2 Karhunen-Loeve Approximations

Any choice of an orthonormal basis in \mathcal{H} , say $(h^k : k \in \mathbb{N})$, yields a L^2 -expansion of a Gaussian process X as (a.s. and L^2 -convergent) sum of the form $X = \sum_{k \in \mathbb{N}} Z_k h^k$ where $Z_k := \tilde{h}^k := \xi(h^k)$ and $h \in \mathcal{H} \mapsto \tilde{h} \in L^2(\Omega)$ is the classical isometry between \mathcal{H} and the Gaussian subspace in $L^2(\Omega)$, sometimes called Paley-Wiener map (see [16], [15], [5, Chapter 3.4] and the appendix). As a reminder that we work with continuous Gaussian processes with the concrete

index set $[0, 1]$, just as for Brownian motion, we shall refer to L^2 -approximation as Karhunen-Loeve (type) approximations³, in the same spirit as we prefer to call \mathcal{H} Cameron-Martin space rather than Reproducing Kernel Hilbert Space.

As in previous sections, let $X = (X^i : i = 1, \dots, d)$ be a centered continuous Gaussian process, with independent components, each with covariance R of finite ρ -variation for some $\rho \in [1, 2)$ and dominated by some 2D control ω . Let \mathbf{X} be the natural lift of X to a $G^3(\mathbb{R}^d)$ -valued process. If $\mathcal{H}_i \subset C([0, 1], \mathbb{R})$ denotes the Cameron-Martin space associated to X^i , the Cameron-Martin space to X is identified with $\oplus_{i=1}^d \mathcal{H}_i$ and if $(h_i^k)_{k \geq 1}$ is an orthonormal basis for \mathcal{H}_i then $\left\{ (h_i^k(\cdot))_{i=1, \dots, d}, k \geq 1 \right\}$ is an orthonormal basis for $\oplus_{i=1}^d \mathcal{H}_i$. We can write $h^k = (h_1^k, \dots, h_d^k)$.

2.1 One Dimensional Estimates

Our object of interest is $X = (X^1, \dots, X^d)$, a d -dimensional real-valued, centered, continuous Gaussian process with independent components. Resultingly, the covariance $R = R(s, t)$ is a diagonal matrix with d entries and in discussing variational regularity of the covariance of a Karhunen-Loeve approximations we may assume that X is in fact 1-dimensional. For any $A \subset \mathbb{N}$ we define

$$\mathcal{F}_A = \sigma(Z_k, k \in A), \quad X_t^A = \mathbb{E}[X_t | \mathcal{F}_A].$$

As in [10], $\omega([a, b] \times [c, d])$ stands for a (2D) control function which controls the ρ -variation of $R(\cdot, \cdot) = \mathbb{E}[X \cdot X_*]$ over the indicated rectangle, i.e.

$$|R|_{\rho\text{-var}; [a, b] \times [c, d]}^\rho \leq \omega([a, b] \times [c, d]).$$

It follows from

$$\mathbb{E}(|X_{s, t}^A|^2) \leq \mathbb{E}(|X_{s, t}|^2) \leq \omega([s, t]^2)^{1/\rho}$$

that X^A can be taken with continuous sample paths. Moreover, X^A is a Gaussian process in its own right and we shall write R^A for its covariance function,

$$R^A(s, t) = \mathbb{E}[X_s^A X_t^A].$$

Lemma 2 *Assume that R is of finite ρ -variation, for some $\rho \geq 1$. Then if $\min\{|A|, |A^c|\} < \infty$*

$$|R^A|_{\rho\text{-var}} < \infty.$$

In particular, if $\rho < 2$, there exists a natural lift of X^A to a $G^3(\mathbb{R}^d)$ -valued process denoted by \mathbf{X}^A .

³Historically, according to a remark in [13], the Karhunen-Loeve approximation corresponds to a specific choice of basis in \mathcal{H} , obtained from the eigenfunction of a (continuous) covariance viewed as integral operator.

Proof. Assume first $|A^c| < \infty$. From [10, Example 7, Proposition 16], using $|h_k|_{\mathcal{H}} = 1$,

$$\begin{aligned} |h_k \otimes h_k|_{\rho\text{-var};[s,t]^2} &\leq |(h_k)|_{p\text{-var};[s,t]}^2 \\ &\leq |R|_{2\text{-var};[s,t]^2}^2. \end{aligned}$$

It follows that

$$\begin{aligned} |R^A|_{p\text{-var};[s,t]^2} &= \left| R - \sum_{k \in A^c} h_k \otimes h_k \right|_{p\text{-var};[s,t]^2} \\ &\leq |R|_{p\text{-var};[s,t]^2} + \sum_{k \in A^c} |h_k \otimes h_k|_{p\text{-var};[s,t]^2} \\ &\leq (1 + m) |R|_{2\text{-var};[s,t]^2}^2. \end{aligned}$$

If A is finite, the proof is even easier. ■

The interest in the above lemma is for $\rho \in [1, 2)$. To obtain uniform estimates valid for all $A \subset \mathbb{N}$, we have to work in 2-variation (but see remark below).

Lemma 3 *Assume that R is of finite 2-variation. Then, the R^A has finite 2-variation, uniformly over all $A \subset \mathbb{N}$. More precisely,*

$$\sup_{A \subset \mathbb{N}} |R^A|_{2\text{-var};[s,t]^2} \leq |R|_{2\text{-var};[s,t]^2}.$$

Proof. Let $D = (t_i)$ a subdivision of $[s, t]$ and set $X_i^A = X_{t_i, t_{i+1}}^A$. Let β be a positive semi-definite symmetric matrix, and let us estimate $\left| \sum_{i,j} \beta_{i,j} \mathbb{E}(X_i^A X_j^A) \right|$. Now

$$\mathbb{E}(X_i^A X_j^A) = \sum_{k \in A} \mathbb{E}(Z_k X_i) \mathbb{E}(Z_k X_j) = \frac{1}{2} \sum_{k \in A} \mathbb{E}((Z_k^2 - \mathbb{E}(Z_k^2)) X_i X_j),$$

so that

$$\sum_{i,j} \beta_{i,j} \mathbb{E}(X_i^A X_j^A) = \frac{1}{2} \sum_{k \in A} \mathbb{E} \left((Z_k^2 - \mathbb{E}(Z_k^2)) \sum_{i,j} \beta_{i,j} X_i X_j \right).$$

As β is symmetric, we can write $\beta = P^T \text{diag}(d_1, \dots, d_{\#D}) P$, with PP^T the identity matrix and (non-negative) eigenvalues (d_i) . By simple linear algebra,

$$\sum_{i,j} \beta_{i,j} X_i X_j = (PX)^T \text{diag}(\dots) (PX) = \sum_i d_i (PX)_i^2.$$

and so

$$\begin{aligned}
\sum_{i,j} \beta_{i,j} \mathbb{E} (X_i^A X_j^A) &= \sum_{k \in A} \sum_i d_i \frac{1}{2} \mathbb{E} \left((Z_k^2 - \mathbb{E} (Z_k^2)) (PX)_i^2 \right) \\
&= \sum_i d_i \sum_{k \in A} \mathbb{E} (Z_k (PX)_i)^2 \\
&\leq \sum_i d_i \mathbb{E} \left((PX)_i^2 \right) \quad (\text{Parseval inequality}) \\
&= \mathbb{E} \left((PX)^T D (PX) \right) \\
&= \sum_{i,j} \beta_{i,j} \mathbb{E} (X_i X_j) \\
&\leq \|\beta\|_{l^2} \|R\|_{2-var} \quad (\text{Hölder inequality})
\end{aligned}$$

Applying this estimate to $\beta_{i,j} = \mathbb{E} (X_i^A X_j^A)$ we find

$$\sqrt{\sum_{i,j} |\mathbb{E} (X_i^A X_j^A)|^2} \leq \|R\|_{2-var}.$$

The proof is finished by taking the supremum over all dissections of $[s, t]$. ■

Remark 4 *The previous proof can easily extend to showing that if R is of finite ρ -variation, where ρ is an integer greater than 2, then for all $A \subset \mathbb{N}$ and $s < t$, $|R^A|_{\rho-var, [s, t]^2} \leq \|R\|_{\rho-var, [s, t]^2}$. This is done by choosing $\beta_{i,j} = \mathbb{E} (X_i^{n,m} X_j^{n,m})^{\rho-1}$ and indeed if $\rho-1 \in \mathbb{N}$ then β is a positive symmetric matrix (this is a simple consequence of Hadamard-Schur's lemma). We could not prove (or disprove) this for general $\rho \geq 1$; with β being defined as fractional Hadamard power,*

$$\mathbb{E} (X_i^{n,m} X_j^{n,m})^{\rho-1} \text{sign} [\mathbb{E} (X_i^{n,m} X_j^{n,m})]$$

If true for $\rho \in [1, 2)$, the present convergence results would have followed directly from Theorem 1.

2.2 Uniform Bounds on the Modulus and Convergence

We now assume that R has finite ρ -variation for some $\rho \in [1, 2)$ dominated by some 2D control ω , and we fix $A \subset \mathbb{N}$, finite or with finite complement, so that X^A admits a natural $G^3(\mathbb{R}^d)$ -valued lift, denoted \mathbf{X}^A . Of course, $\mathbf{X}^{\mathbb{N}} = \mathbf{X}$.

Lemma 5 (Martingale) *For all $s < t$ in $[0, 1]$, the following equality holds in*

$g_3(\mathbb{R}^d)$,

$$\begin{aligned}\mathbb{E}(\ln(\mathbf{X}_{s,t})|\mathcal{F}_A) &= \ln(\mathbf{X}_{s,t}^A) \\ &+ \frac{1}{12} \sum_{i \neq j} X_{s,t}^{A;j} R_{X^{Ac;i}} \left(\begin{matrix} s & s \\ t & t \end{matrix} \right) [e_i, [e_i, e_j]] \\ &- \frac{1}{2} \sum_{i \neq j} \int_s^t R_{X^{Ac;i}} \left(\begin{matrix} u & s \\ t & u \end{matrix} \right) dX_u^{A;j} [e_i, [e_i, e_j]].\end{aligned}$$

(The integral which appears in the last line is a Young-Wiener integral in the sense of [10, Prop. 38]).

Remark 6 Projection to $g_2(\mathbb{R}^d)$ yields to pleasant equality $\mathbb{E}(\ln(\mathbf{X}_{s,t})|\mathcal{F}_A) = \ln(\mathbf{X}_{s,t}^A)$ which explains why martingale arguments [11], [7], [6], [4] are enough to discuss the step-2 case. In contrast, the present lemma shows clearly that martingale arguments are not enough to handle the step-3 case.

Proof. Our proposition at level 1 is $\mathbb{E}(\pi_1(\ln \mathbf{X}_{s,t})|\mathcal{F}_A) = \pi_1(\ln \mathbf{X}_{s,t}^A)$, which is (almost) the definition of X^A . The estimate at level 2 is implied by $\mathbb{E}[\mathbf{X}_{s,t}^{i,j}|\mathcal{F}_A] = (\mathbf{X}^A)_{s,t}^{i,j}$. This is fairly straightforward to prove: one just need to note that conditioning equal L^2 -projection is (trivially) L^2 -continuous and recalling that both \mathbf{X} and \mathbf{X}^A are L^2 -limit of lifted piecewise linear approximations. Level 3 statements is more complicated. We can see as above that for distinct indices i, j, k

$$\mathbb{E}[\mathbf{X}_{s,t}^{i,j,k}|\mathcal{F}_A] = (\mathbf{X}^A)_{s,t}^{i,j,k}.$$

Hence, from [10, Prop. 58, Appendix III], we see that $\mathbb{E}(\ln(\mathbf{X}_{s,t})|\mathcal{F}_A) - \ln(\mathbf{X}_{s,t}^A)$ is equal to

$$\begin{aligned}&\sum_{i \neq j} \mathbb{E} \left(\left\{ \mathbf{X}_{s,t}^{i,i,j} + \frac{1}{12} |X_{s,t}^i|^2 X_{s,t}^j - \frac{1}{2} X_{s,t}^i \mathbf{X}_{s,t}^{i,j} \right\} \middle| \mathcal{F}_A \right) [e_i, [e_i, e_j]] \\ &- \sum_{i \neq j} \left((\mathbf{X}_{s,t}^A)^{i,i,j} + \frac{1}{12} |(X^A)_{s,t}^i|^2 (X^A)_{s,t}^j - \frac{1}{2} (X^A)_{s,t}^i (\mathbf{X}^A)_{s,t}^{i,j} \right) [e_i, [e_i, e_j]].\end{aligned}$$

All the three terms can be written as sums (or L^2 -limits thereof) involving terms of form $X_{r,s}^i X_{t,u}^i X_{v,w}^j$ and since (write $X_{r,s}^i = X_{r,s}^{A;i} + X_{r,s}^{Ac;i}$ and similarly for the other terms)

$$\mathbb{E}(X_{r,s}^i X_{t,u}^i X_{v,w}^j | \mathcal{F}_A) - (X^A)_{r,s}^i (X^A)_{t,u}^i (X^A)_{v,w}^j = X_{v,w}^{A;j} \mathbb{E}(X_{r,s}^{Ac;i} X_{t,u}^{Ac;i}).$$

After integration, we therefore obtain

$$\begin{aligned}\mathbb{E}(\mathbf{X}_{s,t}^{i,i,j} | \mathcal{F}_A) - (\mathbf{X}_{s,t}^A)^{i,i,j} &= \frac{1}{2} \int_s^t \mathbb{E}(|X_{s,u}^{Ac;i}|^2) dX_u^{A;j} \\ &= \frac{1}{2} \int_s^t R_{X^{Ac;i}} \left(\begin{matrix} s & s \\ u & u \end{matrix} \right) dX_u^{A;j},\end{aligned}$$

$$\begin{aligned}
\mathbb{E} \left(|X_{s,t}^i|^2 X_{s,t}^j | \mathcal{F}_A \right) - \left(X_{s,t}^{A;i} \right)^2 X_{s,t}^{A;j} &= X_{s,t}^{A;j} \mathbb{E} \left(|X_{s,t}^{A^c;i}|^2 \right) \\
&= X_{s,t}^{A;j} R_{X^{A^c;i}} \left(\begin{array}{c} s \\ t \end{array}, \begin{array}{c} s \\ t \end{array} \right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E} \left(X_{s,t}^i \mathbf{X}_{s,t}^{i,j} | \mathcal{F}_A \right) - X_{s,t}^{A;i} \mathbf{X}_{s,t}^{A;j} &= \int_s^t \mathbb{E} \left(X_{s,t}^{A^c;i} X_{s,u}^{A^c;i} \right) dX_u^{A;j} \\
&= \int_s^t R_{X^{A^c;i}} \left(\begin{array}{c} s \\ t \end{array}, \begin{array}{c} s \\ u \end{array} \right) dX_u^{A;j}
\end{aligned}$$

That concludes the proof. ■

Proposition 7 For all $s < t, A, i \neq j$, for some constant C ,

$$\left| \int_s^t R_{X^{A^c;i}} \left(\begin{array}{c} u \\ t \end{array}, \begin{array}{c} s \\ u \end{array} \right) dX_u^{A;j} \right|_{L^2}^2 \leq C \omega \left([s, t]^2 \right)^{3/\rho}.$$

Proof. From

$$\int_s^t R_{X^{A^c;i}} \left(\begin{array}{c} u \\ t \end{array}, \begin{array}{c} s \\ u \end{array} \right) dX_u^{A;j} = \mathbb{E} \left(\int_s^t R_{X^{A^c;i}} \left(\begin{array}{c} u \\ t \end{array}, \begin{array}{c} s \\ u \end{array} \right) dX_u^j \middle| \mathcal{F}_A \right)$$

it suffices to consider the integral with integrator dX^j . We define

$$f(u) := R_{X^{A^c;i}} \left(\begin{array}{c} u \\ t \end{array}, \begin{array}{c} s \\ u \end{array} \right).$$

and note that $f(s) = 0$. It is easy to see that for $u < v$ in $[s, t]$,

$$|f_{u,v}|^2 \leq |R_{X^{A^c;i}}|_{2-var;[u,v] \times [s,t]}^2 + |R_{X^{A^c;i}}|_{2-var;[s,t] \times [u,v]}^2.$$

Noting super-additivity of the right hand side in $[u, v]$ and using Lemma 3,

$$\begin{aligned}
|f|_{2-var;[s,t]}^2 &\leq 2 |R_{X^{A^c;i}}|_{2-var;[s,t]}^2 \\
&\leq 2 |R_{X^i}|_{2-var;[s,t]}^2 \\
&\leq 2 \omega \left([s, t]^2 \right)^{2/\rho}.
\end{aligned}$$

Now, f has finite 2-variation and the covariance of the integrator dX^j has finite ρ -variation, $\rho \in [1, 2)$ controlled by ω . Thanks to $1/2 + 1/\rho > 1$ we can conclude with the "Young-Wiener" estimate [10, Prop. 38]. ■

Putting the last two results together and using [10, Proposition 24], we obtain the following theorem.

Theorem 8 For all $s < t$ in $[0, 1]$ there exists $C = C(\rho)$ such that

$$\sup_{A \subset \mathbb{N}, \min\{|A|, |A^C|\} < \infty} \mathbb{E} \left(\|\mathbf{X}_{s,t}^A\|^2 \right) \leq C \omega \left([s, t]^2 \right)^{\frac{1}{\rho}}.$$

For $p > 2\rho$ and $\omega \left([0, 1]^2 \right) \leq K$ there exists $\eta = \eta(p, \rho, K) > 0$ such that

$$\sup_{A \subset \mathbb{N}, \min\{|A|, |A^C|\} < \infty} \mathbb{E} \left(\exp \eta \|\mathbf{X}^A\|_{p-var; [0,1]}^2 \right) < \infty.$$

If $\omega \left([s, t]^2 \right) \leq K |t - s|$ for all $s < t$ in $[0, 1]$ we can replace $\|\mathbf{X}^A\|_{p-var; [0,1]}$ by $\|\mathbf{X}^A\|_{1/p-H\ddot{o}l; [0,1]}$.

We now discuss convergence results.

Theorem 9 Let $A_n = \{1, \dots, n\}$. For any $p > 2\rho$ and $q \in [1, \infty)$,

$$d_{p-var; [0,1]}(\mathbf{X}^{A_n}, \mathbf{X}) \rightarrow 0 \text{ in } L^q(\Omega) \text{ as } n \rightarrow \infty, \quad (2)$$

$$\left\| \mathbf{X}^{A_n^c} \right\|_{p-var; [0,1]} \rightarrow 0 \text{ in } L^q(\Omega) \text{ as } n \rightarrow \infty. \quad (3)$$

If ω is Hölder dominated, i.e. $\sup_{0 \leq s < t \leq 1} \omega \left([s, t]^2 \right) / |t - s|^{1/p} < \infty$, then

$$d_{1/p-H\ddot{o}l; [0,1]}(\mathbf{X}^{A_n}, \mathbf{X}) \rightarrow 0 \text{ in } L^q(\Omega) \text{ as } n \rightarrow \infty, \quad (4)$$

$$\left\| \mathbf{X}^{A_n^c} \right\|_{1/p-H\ddot{o}l; [0,1]} \rightarrow 0 \text{ in } L^q(\Omega) \text{ as } n \rightarrow \infty. \quad (5)$$

Proof. Ad (2), (4): From [10, appendix I] and Theorem 8, it is enough to prove that for any fixed $t \in [0, 1]$,

$$d \left(\mathbf{X}_t^{A_n}, \mathbf{X}_t \right) \rightarrow 0$$

in L^q or, in fact, in probability (thanks to the uniform L^q -bounds for all $q < \infty$ in Theorem 8). The topology induced by d on $G^3(\mathbb{R}^d)$ is consistent with the manifold topology $G^3(\mathbb{R}^d) \subset T^3(\mathbb{R}^d)$ and in particular with the topology induced from the Euclidean structure on $g^3(\mathbb{R}^d) = \ln(G^3(\mathbb{R}^d))$, seen as global chart for $G^3(\mathbb{R}^d)$. It is therefore enough to show for $N = 1, 2, 3$ we have pointwise convergence,

$$\pi_N \left(\ln \left(\mathbf{X}_t^{A_n} \right) - \ln \left(\mathbf{X}_t \right) \right) \rightarrow 0 \text{ in probability.}$$

By martingale convergence, this is obvious for $N = 1, 2$ but for $N = 3$ we have to handle the correction which we identified in Lemma 5,

$$\left(\frac{1}{12} X_{s,t}^{A; j} R_{X^{A^c; i}} \begin{pmatrix} s & s \\ t & t \end{pmatrix} - \frac{1}{2} \int_s^t R_{X^{A^c; i}} \begin{pmatrix} u & s \\ t & u \end{pmatrix} dX_u^{A; j} \right) [e_i, [e_i, e_j]].$$

All we need is pointwise convergence in probability to zero of this expression. Clearly, $X^{A_n^c} = \mathbb{E} [X | \mathcal{F}_{\{n+1, n+2, \dots\}}] \rightarrow 0$ a.s. and in all L^q as $n \rightarrow \infty$. It follows that $R_{X^{A_n^c}; i} \rightarrow 0$ pointwise which takes care of the first summand. The second term is a Young-Wiener integral in the sense of [10, Proposition 38]. From our uniform estimates and interpolation, $R_{X^{A_n^c}; i} \rightarrow 0$ in $(2 + \varepsilon)$ -variation. Using notation from the last proposition,

$$\int_s^t f(u) dX_u^{A_n, j} = \mathbb{E} \left(\int_s^t f(u) dX_u^j \middle| \mathcal{F}_{A_n} \right),$$

and it is enough to show that $\int_s^t f(u) dX_u^j \rightarrow 0$ in L^2 . Now,

$$|f|_{(2+\varepsilon)\text{-var}; [s, t]}^{2+\varepsilon} \leq C |R_{X^{A_n^c}; i}|_{(2+\varepsilon)\text{-var}; [s, t]}^{2+\varepsilon} \rightarrow 0$$

and using the Young-Wiener estimate for ε chosen small enough (namely such that $(2 + \varepsilon)^{-1} + \rho^{-1} > 1$ which is always possible since $\rho \in [1, 2)$) we obtain the required convergence in L^2 and hence in probability as required.

Ad (3), (5): As in the first part of the proof, it is enough to show that, for fixed $t \in [0, 1]$, $\mathbf{X}_t^{A_n^c} \rightarrow 0$ in probability or, equivalently,

$$\ln \left(\mathbf{X}_t^{A_n^c} \right) \rightarrow 0 \text{ in probability.}$$

We first claim that $\mathbb{E} (\ln (\mathbf{X}_t) | \mathcal{F}_{A_n^c}) \rightarrow 0$. Indeed, by backward martingale convergence and Kolmogorov's 0-1 law,

$$\begin{aligned} \mathbb{E} (\ln (\mathbf{X}_t) | \mathcal{F}_{A_n^c}) &\rightarrow \mathbb{E} (\ln (\mathbf{X}_t) | \cap_k \mathcal{F}_{A_k^c}) \text{ a.s. and in all } L^q \\ &\stackrel{\text{a.s.}}{=} \mathbb{E} (\ln \mathbf{X}_t) = 0. \end{aligned}$$

Let us detail why the $(g^3(\mathbb{R}^d))$ -valued expectation of $\ln \mathbf{X}_t$ is indeed 0. The only interesting case is projection to the level $N = 3$. By the expansion of $\ln \mathbf{X}_t$ given in [10, Prop. 58] we know that $\pi_3(\ln \mathbf{X}_t)$ involves precisely terms of form $\mathbf{X}_t^{i, j, k}, \mathbf{X}_t^{i, i, j}, (\mathbf{X}_t^i)^2 \mathbf{X}_t^j$ and $\mathbf{X}_t^i \mathbf{X}_t^{i, j}$ (with disjoint indices $i, j, k \in \{1, \dots, d\}$). Then $\mathbb{E} \mathbf{X}_t^{i, j, k} = 0$ since $\mathbf{X}_t^{i, j, k}$ is an element of the third homogenous Wiener-Itô chaos. $\mathbb{E} \mathbf{X}_t^{i, i, j} = 0$ since an approximation argument reduces this to the case of nice sample paths in which case the assertion is clear using Fubini and $\mathbb{E} \mathbf{X}^j = \mathbb{E} X^j = 0$. Similar (but easier) for $(\mathbf{X}_t^i)^2 \mathbf{X}_t^j$. At last $\mathbb{E} (\mathbf{X}_t^i \mathbf{X}_t^{i, j}) = 0$ by orthogonality of the first and second Wiener-Itô chaos.) The proof will be finished if we can handle the difference between $\ln (\mathbf{X}_t^{A_n^c})$ and $\mathbb{E} (\ln (\mathbf{X}_t) | \mathcal{F}_{A_n^c})$. But using Lemma 5, this is done in the same way as in the first part of the proof. ■

3 Support Theorem for the Law of X

We recall the standing assumptions. $X = (X^i : i = 1, \dots, d)$ is a centered continuous Gaussian process on $[0, 1]$, with independent components and finite

covariance of finite $\rho \in [1, 2)$ -variation, dominated by some 2D control ω . From [10, Theorem 35] we know that, for $p \in (2\rho, 4)$, X lifts to a (random) geometric p -rough path \mathbf{X} with a.e. sample path in $C_0^{0,p-var}([0, 1], G^3(\mathbb{R}^d))$. If ω is Hölder dominated we have sample paths in $C_0^{0,1/p-Höl}([0, 1], G^3(\mathbb{R}^d))$.

Theorem 10 *Let $\mathbb{P}_*\mathbf{X}$ denote the law of \mathbf{X} , a Borel measure on the Polish space $C_0^{0,p-var}([0, 1], G^3(\mathbb{R}^d))$. Then*

$$\text{supp}[\mathbb{P}_*\mathbf{X}] = \overline{S_3(\mathcal{H})}$$

where support and closure are with respect to p -variation topology. If ω is Hölder dominated, $\omega([s, t]^2) \leq K|t - s|$ for some constant K , we can use $1/p$ -Hölder topology instead of p -variation topology.

Remark 11 *Note that $S_3(\mathcal{H})$ is canonically defined by iterated Young integration. Indeed, any $h \in \mathcal{H}$ has finite ρ -variation ([10, Proposition 16] and, under the standing assumption of $\rho \in [1, 2)$, lifts to a $G^3(\mathbb{R}^d)$ -valued paths (of finite ρ -variation) by iterated Young integration.*

Proof. Applying theorem 9 with $\mathbf{X}^{\{n,n+1,\dots\}}$ instead of $\mathbf{X} = \mathbf{X}^{\{1,2,3,\dots\}}$ readily shows that for all $n \in \{1, 2, 3, \dots\}$,

$$d_{p-var; [0, 1]}(\mathbf{X}^{\{n,n+1,\dots,m\}}, \mathbf{X}^{\{n,n+1,\dots\}}) \rightarrow_{m \rightarrow \infty} 0 \text{ in probability}$$

and

$$\left\| \mathbf{X}^{\{m+1,m+2,\dots\}} \right\|_{p-var; [0, 1]} \rightarrow_{m \rightarrow \infty} 0.$$

It is also clear⁴ that $\omega \mapsto \mathbf{X}(\omega)$ restricted to \mathcal{H} coincides with $S_3(\cdot)$ and is continuous thanks to $\mathcal{H} \hookrightarrow C^{\rho-var}$ and basic continuity properties of Young integration. We can then conclude with a support description for abstract Wiener functionals give in the appendix (proposition 13). ■

Remark 12 *The idea of proposition 13 is to find at least one $\omega \in C([0, 1], \mathbb{R}^d)$ such that $\mathbf{X}(\omega) \in \text{supp}[\mathbb{P}_*\mathbf{X}]$ and such that there exists a sequence $(g_n) \subset \mathcal{H}$ such that $\mathbf{X}(\omega - g_n)$ converges with respect to d_{p-var} to $\mathbf{X}(0) \equiv \exp(0) \in G^3(\mathbb{R}^d)$. If elements of \mathcal{H} (or at least some ONB $(h_n) \subset \mathcal{H}$) have good enough regularity, namely finite q -variation with $1/p + 1/q > 1$, then this is relatively easy: One can take*

$$g_n(\cdot, \omega) = \sum_{i=1}^n \xi(h_i) |_{\omega} h_i(\cdot) \in C^{q-var};$$

recalling that $\mathbf{X}(\omega)$ was constructed as limit (in probability) of piecewise linear approximations⁵ it follows by basic properties of the rough path translation

⁴After all, $\mathbf{X}(\omega)$ was defined as the the limit in probability of piecewise linear approximations.

⁵On the null-set on which piecewise linear approximations do not converge, $\mathbf{X}(\omega)$ is defined as an arbitrary constant.

operator (using crucially $1/p + 1/q > 1$) that for almost every ω

$$\begin{aligned} \mathbf{X}(\omega - g_n) &= T_{-g_n} \mathbf{X}(\omega) = \lim_{m \rightarrow \infty} T_{-g_n} \mathbf{X}^{\{1, \dots, m\}}(\omega) \\ &= \lim_{m \rightarrow \infty} \mathbf{X}^{\{n+1, \dots, m\}}(\omega) \\ &= \mathbf{X}^{\{n+1, n+2, \dots\}}(\omega) \end{aligned}$$

and by (3) this converges indeed to $\mathbf{X}(0) \equiv \exp(0) \in G^3(\mathbb{R}^d)$. This argument (construction of a nice ONB in \mathcal{H}) was used by [6] in the context of fractional Brownian motion, using the particular structure of its Volterra kernel. (In fact, as long as Hurst parameter $H > 1/4$, all Cameron-Martin paths have sufficient variational regularity, cf. [9]).

Unfortunately, these arguments breaks down in the general setting when $\rho \geq 3/2$, since $p = 2\rho + \varepsilon$, that is, when dealing with general Gaussian rough paths of p -variation regularity with $p \geq 3$. Our proposition 13 is based on a careful revision of the above; it avoids any use of translation operators and is presented in its natural generality: the setting of abstract Wiener functionals. Support descriptions of abstract Wiener functionals were studied by Aida-Kusuoka-Stroock and it seems worthwhile to point out that their abstract support theorem [1, Cor 1.13] applies to our situation when $\rho < 3/2$ but not beyond. More precisely, only for $\rho < 3/2$ can we translate \mathbf{X} deterministically in Cameron-Martin directions: then $\omega \mapsto \mathbf{X}(\omega)$ is automatically \mathcal{K} -continuous in the sense of Aida, Kusuoka and Stroock and the support description follows from their results. However, their notion of \mathcal{K} -continuity does not cover the regime $\rho \in [3/2, 2)$; at least not without addition information (such as the existence of an ONB of \mathcal{H} contained in C^{q-var} with $1/p + 1/q > 1 \dots$).

We recall that such a support description in rough path topology yields, as consequence of Lyons' limit theorem [17] and without further work, a Stroock-Varadhan type support theorem for solutions of rough path differential equations (RDEs) driven by the (random) geometric p -rough path \mathbf{X} . For Brownian motion, this application was first carried out by Ledoux et al. [14]; they also conjecture the extension to fractional Brownian motion which was then obtained for $H > 1/3$ (i.e. without third iterated integrals) in [8] and [6]. It is well-known that the Cameron-Martin space \mathcal{H}^H of d -dimensional fractional Brownian motion started at 0 contains $C_0^1([0, 1], \mathbb{R}^d)$. From [10, Theorem 35], ω is Hölder dominated and we may work in α -Hölder topology for $\alpha \in [0, H)$, in particular

$$\overline{S_3(\mathcal{H}^H)} = C_0^{0, \alpha-H\ddot{o}l}([0, 1], G^3(\mathbb{R}^d)).$$

A natural lift of fractional Brownian Motion (also known as *enhanced fractional Brownian motion*) exists for $H > 1/4$ and we see that its support is full, i.e. equals $C_0^{0, \alpha-H\ddot{o}l}([0, 1], G^3(\mathbb{R}^d))$ for any $\alpha \in [0, H)$ and $H > 1/4$. Remark that the extension to d independent fractional Brownian motions with different Hurst indices greater than $1/4$ is immediate.

4 Appendix

Let μ be a mean zero Gaussian measure on a real separable Banach space E . Following the standard references [15] and [5, Chapter 4], there is an abstract Wiener space factorization of the form

$$E^* \xrightarrow{\iota^*} L^2(\mu) \xrightarrow{\iota} E.$$

Here ι^* denotes the embedding of E^* into $L^2(\mu)$; the notation is justified because ι^* can be identified as the adjoint of the map ι whose construction we now recall: Let (K_n) be a compact exhaustion of E . If $\varphi \in L^2(\mu)$, then $\iota(\varphi I_{K_n})$ can be identified with the (strong) expectation

$$\int_{K_n} x \varphi(x) d\mu(x).$$

This is a Cauchy sequence in E and we denote the limit by $\iota(\varphi)$. One defines E_2^* to be the closure of E^* , or more precisely: $\iota^*(E^*)$, in $L^2(\mu)$. The reproducing kernel Hilbert space \mathcal{H} of μ is then defined as

$$\mathcal{H} := \iota(E_2^*) \subset \iota(L^2(\mu)) \subset E.$$

The map ι restricted to E_2^* is linear and bijective onto \mathcal{H} and induces a Hilbert structure

$$\langle h, g \rangle_{\mathcal{H}} := \langle \tilde{h}, \tilde{g} \rangle_{L^2(\mu)} \quad \forall h, g \in \mathcal{H}$$

where we set $\tilde{h} \equiv (\iota|_{E_2^*})^{-1}(h)$, of course the meaning of \tilde{g} is similar. To summarize, we have the picture

$$\begin{aligned} E^* \xrightarrow{\iota^*} \iota^*(E^*) &\subset \overline{\iota^*(E^*)} =: E_2^* \subset L^2(\mu) \xrightarrow{\iota} E. \\ \text{and } \iota|_{E_2^*} &: E_2^* \longleftrightarrow \mathcal{H} \subset E. \end{aligned}$$

Under μ , the map $x \mapsto \tilde{h}(x)$ is a Gaussian random variable with variance $|h|_{\mathcal{H}}^2$. We can think of $x \mapsto X(x) = x$ as E -valued random variable with law μ . Given an ONB (h_k) in \mathcal{H} we have the L^2 -expansion

$$X(x) = \lim_{m \rightarrow \infty} \sum_{k=1}^m \tilde{h}_k(x) h_k \text{ a.s.}$$

where the sum converges in E for μ -a.e. x and in all $L^p(\mu)$ -spaces, $p < \infty$. For any $A \subset \mathbb{N}$ define⁶

$$X^A(x) = \sum_{k \in A} \tilde{h}_k(x) h_k.$$

⁶If $|A| < \infty$, this is a finite sum with values in \mathcal{H} ; if $|A| = \infty$ this sum converges in E for μ -almost every x and in every $L^p(\mu)$. All this follows from X^A being the conditional expectation of X given $\{h_k : k \in A\}$.

Proposition 13 Consider a L^2 -expansion of $X(x)$ with respect to an ONB (h_k) in \mathcal{H} is such that

$$\tilde{h}_k \equiv (\iota|_{E_2^*})^{-1}(h_k) \subset \iota^*(E^*)$$

(rather than $\overline{\iota^*(E^*)}$, this is always possible; we think of \tilde{h}_k as element in E^* .) Assume (M, ρ) is a Polish space equipped with its Borel σ -algebra \mathcal{M} and $\varphi : (E, \mathcal{B}) \rightarrow (M, \mathcal{M})$ is a measurable map such that⁷ is approximately continuous in the sense that for all $n \in \{1, 2, 3, \dots\}$,

$$\rho\left(\varphi\left(X^{\{n, \dots, m\}}\right), \varphi\left(X^{\{n, \dots\}}\right)\right) \rightarrow_{m \rightarrow \infty} 0 \text{ in probability}$$

and

$$\rho\left(\varphi\left(X^{\{m+1, \dots\}}\right), \varphi(0)\right) \rightarrow_{m \rightarrow \infty} 0 \text{ in probability.}$$

Assume furthermore that the restriction of $\varphi|_{\mathcal{H}} : \mathcal{H} \rightarrow M$ is continuous. Then the topological support of $\varphi_*\mu$ is equal to the closure of $\varphi(\mathcal{H})$.

Proof. Observing that $X^{\{1, \dots, n\}}$ is a random but finite linear combination of elements in \mathcal{H} , the inclusion

$$\text{supp}(\varphi_*\mu) \subset \varphi(\mathcal{H})$$

is clear. The idea for the converse is to work with

$$\hat{\varphi}(x) := \lim_{m \rightarrow \infty} \varphi\left(X^{\{1, \dots, m\}}(x)\right).$$

Observe that $\hat{\varphi} \equiv \varphi$ on \mathcal{H} . Indeed, this from continuity of $\varphi|_{\mathcal{H}} : \mathcal{H} \rightarrow M$ together with

$$X^{\{1, \dots, m\}}(h) = \sum_{k=1}^m \tilde{h}_k(h) h_k = \sum_{k=1}^m \langle h_k, h \rangle_{\mathcal{H}} h_k,$$

which plainly converges in \mathcal{H} to h as $m \rightarrow \infty$. Moreover, by the first convergence assumption $\hat{\varphi}(x) = \varphi(X(x)) = \varphi(x)$ for μ -a.e. x . We shall prove in the next lemma that there exists at least one (fixed) element $x \in X$ such that $\hat{\varphi}(x) \in \text{supp}(\varphi_*\mu) = \text{supp}(\hat{\varphi}_*\mu)$ and such that there exists $(g_n) \subset H$, which may and will depend on x , such that

$$\hat{\varphi}(x - g_n) \rightarrow \hat{\varphi}(0) \text{ as } n \rightarrow \infty.$$

It then follows from the Cameron-Martin theorem, that $\hat{\varphi}(x - g_n)$ and any limit point such as $\hat{\varphi}(0)$ is contained in the support. Again, using Cameron-Martin, any $\hat{\varphi}(h)$ with $h \in \mathcal{H}$ will be in the support and this finishes the converse conclusion. ■

⁷Converges in probability means converges in measure with respect to μ .

Lemma 14 *Under the assumptions of the previous proposition,*

$$\mu \{x : \exists (g_n) \subset \mathcal{H} : \hat{\varphi}(x - g_n) \rightarrow_{n \rightarrow \infty} \varphi(0)\} = 1.$$

(Trivially, $\mu \{x : \hat{\varphi}(x) \in \text{supp}(\hat{\varphi}_* \mu)\} = 1$ and the intersection of these sets has also full measure).

Proof. By assumption, $h_i = \iota(\tilde{h}_i)$ is an ONB in \mathcal{H} and all \tilde{h}_i are of form $\langle \lambda_i, \cdot \rangle_{E^*, E}$. As in [5, (3.4.14)],

$$\begin{aligned} \langle \lambda_i, h_j \rangle_{E^*, E} &= \left\langle \langle \lambda_i, \cdot \rangle_{E^*, E}, \tilde{h}_j \right\rangle_{L^2(\mu)} \\ &= \left\langle \tilde{h}_i, \tilde{h}_j \right\rangle_{L^2(\mu)} = \langle h_i, h_j \rangle_{\mathcal{H}}. \end{aligned}$$

Thinking of \tilde{h}_i as elements on E^* , we abuse of notation and write $\langle \tilde{h}_i, h_j \rangle_{E^*, E}$ instead of $\langle \lambda_i, h_j \rangle_{E, E^*}$. What we have just seen is that

$$\langle \tilde{h}_i, h_j \rangle_{E^*, E} = \delta_{i,j}$$

where $\delta_{i,j} = 1$ if $i = j$ and zero otherwise. Let us fix n . By definition,

$$\forall x \in E : X^{\{1, \dots, m\}}(x) := \sum_{i=1}^m \langle \tilde{h}_i, x \rangle_{E^*, E} h_i$$

and using linearity of the pairing between E^* and E we find that, for $m > n$,

$$\forall x \in E : X^{\{1, \dots, m\}} \left(x - \sum_{j=1}^n \langle \tilde{h}_j, x \rangle_{E^*, E} h_j \right) = \sum_{i=n+1}^m \langle \tilde{h}_i, x \rangle_{E^*, E} h_i \quad (6)$$

and so, again for all $x \in E$,

$$\varphi \left(X^{\{1, \dots, m\}} \left(x - \sum_{j=1}^n \langle \tilde{h}_j, x \rangle_{E^*, E} h_j \right) \right) = \varphi \left(\sum_{i=n+1}^m \langle \tilde{h}_i, x \rangle_{E^*, E} h_i \right). \quad (7)$$

By assumption, for every fixed $n \in \mathbb{N}$, the right hand converges in probability to $\varphi(X^{\{n+1, \dots\}}(x))$. But then the left hand side must also converges (in probability) and by a.s. uniqueness of limits in probability,

$$\begin{aligned} \lim_{m \rightarrow \infty} \varphi \left(X^{\{1, \dots, m\}} \left(x - \sum_{j=1}^n \langle \tilde{h}_j, x \rangle_{E^*, E} h_j \right) \right) &= \lim_{m \rightarrow \infty} \varphi \left(\sum_{i=n+1}^m \langle \tilde{h}_i, x \rangle_{E^*, E} h_i \right) \\ &= \varphi \left(X^{\{n+1, \dots\}}(x) \right) \end{aligned}$$

for μ -a.e. $x \in E$. The countable union of nullsets still being a nullset it follows that (i) the set

$$\left\{ x : \forall n \in \mathbb{N}: \lim_{m \rightarrow \infty} \varphi \left(X^{\{1, \dots, m\}} \left(x - \sum_{j=1}^n \langle \tilde{h}_j, x \rangle_{E^*, E} h_j \right) = \varphi \left(X^{\{n+1, \dots\}}(x) \right) \right) \right\}$$

has full μ -measure. By definition of $\hat{\varphi}$ and our assumption, $\varphi(X^{\{n+1, \dots\}}(x)) \rightarrow \varphi(0) = \hat{\varphi}(0)$ in probability, and hence for a.e. x along a subsequence (n_l) , we see that

$$\mu \left\{ x : \hat{\varphi} \left(x - \sum_{j=1}^{n_l} \langle \tilde{h}_j, x \rangle_{E^*, E} h_j \right) \rightarrow_{l \rightarrow \infty} \hat{\varphi}(0) \right\} = 1$$

The proof is finished by defining $(g^l) \subset \mathcal{H}$ by $g^l = \sum_{j=1}^{n_l} \langle \tilde{h}_j, x \rangle_{E^*, E} h_j$. ■

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